Sparse sampling in MRI:
From basic theory to clinical application

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Objective

• Provide an intuitive overview of compressed sensing as applied to MRI
Objective

- Provide an intuitive overview of compressed sensing as applied to MRI
- This is not a research talk
  - I will use my data to illustrate points
- This is an emerging technology
  - There are unanswered questions
  - Some of this talk will be speculative
Outline

• MR Physics

• k-space and sampling requirements

• Compressed sensing
  • Constrained reconstruction, sparsity, and random sampling

• Applications

• Neuroimaging
Magnetic Resonance Imaging

Main magnet, ~1 Tesla
(magnetize the sample)

Gradient coils, mT/m
(spatial localization)

Radio-frequency coil(s), ~uT
(transmission and reception)
Magnetic Resonance Imaging

- Echo time
- Repetition time

Radio-frequency

Readout

Phase encode

Slice select

Spatial encoding gradients
Magnetic Resonance Imaging

- Non invasive
- High resolution
- Multiple intrinsic contrast mechanisms
- Arbitrary slice orientation
Magnetic Resonance Imaging

Acquisition time

**Spatial encoding**
- Serially acquire all of the points in an image
- Solutions
  - Adjust resolution and field-of-view to require fewer points
  - Undersample the data

**Low sensitivity**
- Very few spins contribute signal; lots contribute noise
- Solutions
  - Adjust resolution
  - Scan longer
Magnetic Resonance Imaging

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**Undersample without noise amplification**
Magnetic Resonance Imaging

Sampling requirements

$k$-space* → image-space

\[ \text{FOV}_{ky} \]

\[ \text{FOV}_y = 1/\text{dk}_y \]

\[ \text{dy} = 1/\text{FOV}_{ky} \]

*All $k$-space images are logarithmically scaled
Magnetic Resonance Imaging

Sampling requirements

$k$-space*

Image-space

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$\text{dy} = 1/\text{FOV}_k y$

*All $k$-space images are logarithmically scaled
Magnetic Resonance Imaging

Sampling requirements

$k$-space*

$\text{FOV}_{ky}$

$dk_y$

$\text{FOV}_{ky}$

$\text{FOV}_y$

$dy$

$\text{FOV}_y = 1/dk_y$

$dy = 1/\text{FOV}_{ky}$

*All k-space images are logarithmically scaled
Magnetic Resonance Imaging
Undersampling

\[ dk_x \]
\[ dk_y \]

FOV_x

Fourier Transform
Magnetic Resonance Imaging

Undersampling

\( dk_x \)

\( dk_y \)

sub-Nyquist sampling
Magnetic Resonance Imaging

Undersampling

\[ a_1 = S_{1,m} \ p_m + S_{1,n} \ p_n \]

Aliased signal

Coil sensitivity at ‘m’

\[ a_1 \]

Coil sensitivity at ‘n’

True MRI signal at ‘m’

True MRI signal at ‘n’

True image
Magnetic Resonance Imaging
Parallel Imaging

\[ a_1 = S_{1,m} p_m + S_{1,n} p_n \]

\[ a_2 = S_{2,m} p_m + S_{2,n} p_n \]

In matrix form:

\[ [A] = [S][P] \]

...with solution:

\[ P = (S*S)^{-1}S*A \]

SENSE: Sensitivity Encoding
Pruessmann KP, et al. MRM 1999
Magnetic Resonance Imaging

Parallel Imaging

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Magnetic Resonance Imaging
Parallel Imaging

- Acceleration rate $\leq$ number of receiver coils
- Receiver coils must have a unique sensitivity along the accelerated dimensions
Recap

- MRI is insensitive and very slow
- Data is sampled in k-space
- Uniform undersampling produces coherent aliasing
- Unwrapping is possible with multiple receiver coils (parallel imaging)
Random Undersampling

Fourier Transform

Fourier Transform
Random Undersampling

Fourier Transform

Fourier Transform
Random Undersampling

- The question is: how do we preserve the incoherent aliasing and fill in the missing data?

- Answer: context
Constrained Reconstruction

Fourier Transform
Constrained Reconstruction

Fourier Transform

mask
Constrained Reconstruction

Fourier Transform

mask
Constrained Reconstruction

Fourier Transform

Replace

Fourier Transform
Constrained Reconstruction

Fourier Transform

Replace

Fourier Transform

Fourier Transform
Constrained Reconstruction

Fourier Transform

Replace

Fourier Transform

mask
Constrained Reconstruction

Iterate...

Fourier Transform

Fourier Transform
Constrained Reconstruction

- Initial observations:
  - Promote expected characteristic
  - Enforce data consistency
  - Iterate to improve the image
Mathematical Model

\[
\begin{align*}
\text{minimize} & \quad \| \Psi m \|_0 \\
\text{subject to} & \quad \| Fm - y \|_2^2 < \epsilon
\end{align*}
\]

\{ \text{Sparsity} \}
\{ \text{Data consistency} \}

- Two types of norms:
  a) $l_0$: counts the number of coefficients
  b) $l_2$: sum-of-squares

- $\Psi$: Sparsity Transform
- $m$: Image
- $F$: Fourier Transform
- $y$: Acquired k-space data
- $\epsilon$: Noise threshold
Mathematical Model

\[
\begin{align*}
\text{minimize} \quad & \| \Psi m \|_0 \\
\text{subject to} \quad & \| F m - y \|_2^2 < \epsilon
\end{align*}
\]

Choose an image that:

a) (mostly) conforms to the image model
b) is consistent with the acquired data
Transform Sparsity

- A mathematical operation that allows the image to be described with a small number of coefficients
Transform Sparsity

• Images are ‘natural’ (adjacent pixels are often correlated)

• Wavelet Transform $\Psi_m$

Image domain

Wavelet domain
Transform Sparsity

- Images are ‘natural’ (adjacent pixels are often correlated)
- Wavelet Transform $\Psi m$

Image domain          Wavelet domain
Transform Sparsity

- Images are ‘natural’ (adjacent pixels are often correlated)
- Wavelet Transform $\Psi m$

Fully sampled

Undersampled
Transform Sparsity

- Dynamic images do not change very much

Time frame: 1 2 3 4

Finite difference in time
Finite differences

\[(T - I)^m\]

- Time shift
- Identity

Dynamic images do not change very much

Finite differences
Mathematical Model

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\( \Psi \) Sparsity Transform \quad \( m \) Image \quad \( F \) Fourier Transform \quad \( y \) Acquired k-space data \quad \( \epsilon \) Noise threshold

- Choose an image that:
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Mathematical Model

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\end{align*}
\]

- Sparsity
- Data consistency

- This problem is very difficult to solve
- It is non-convex (may have local minima)
Mathematical Model

$$\min_m \left\{ ||Fm - y||^2_2 + \lambda ||\Psi m||_1 \right\}$$

- Convert the $l_0$ norm to an $l_1$ norm
- Sum of absolute values
- Still promotes sparsity
Mathematical Model

\[ \min_m \left\{ \|Fm - y\|_2^2 + \lambda \, \|\Psi m\|_1 \right\} \]

- This problem is convex (no local minima)
- Conjugate gradient
- Thresholding techniques
- Solution is iterative (very slow!)
- Must choose \( \lambda \)

\( \Psi \) Sparsity Transform
\( m \) Image
\( F \) Fourier Transform
\( y \) Acquired k-space data
\( \epsilon \) Noise threshold
\( \lambda \) Regularization factor
Regularization factor

\[
\min_m \left\{ \| Fm - y \|_2^2 + \lambda \| \Psi m \|_1 \right\}
\]

\(\lambda\) too small? \(\lambda\) just right? \(\lambda\) too big?
Reconstruction Movie

k-space  image-space  wavelet-space
Sampling Patterns

- Avoid coherent aliasing in sparsity domain
  - Pseudo-random sampling
- Undersample in domains/dimensions that take time to acquire
  - Often phase-encode dimensions in k-space
- Undersample in as many dimensions as possible

2D $\rightarrow$ 1D undersampling

3D $\rightarrow$ 2D undersampling

\[
\begin{align*}
\text{k}_x & \quad \text{k}_y \\
\text{(in/out of screen)} & \quad \text{k}_z
\end{align*}
\]
Sampling Patterns

- Avoid coherent aliasing in sparsity domain
  - Pseudo-random sampling
  - Can test several different patterns

M. Lustig et al. MRM 2007
Sampling Patterns

- The center of k-space (describes the low resolution image) has a lot of energy
- It should be sampled more heavily

Not sparse!
Sampling Patterns

- Variable density sampling compensates for the lack of sparsity at low resolutions
Compressed Sensing and Parallel Imaging

$l_1$-SPIRiT

(Iterative Self-consistent Parallel Imaging Reconstruction)
Lustig M. MRM 2010

$$\min_m \left\{ \|Fm - y\|_2^2 + \lambda_0 \|(G - I)m\|_2 + \lambda_1 \|\Psi m\|_1 \right\}$$

CS-SENSE

$$\min_m \left\{ \|FSm - y\|_2^2 + \lambda \|\Psi m\|_1 \right\}$$
Recent work from MREL

- We have been working to implement compressed sensing and parallel imaging on the scanner
- GE 3T scanners at HSC
- Fat/water separation
- Visualization of the upper airway
- Neuroimaging
Phantom images

1.0x

2.0x

8.0x
Phantom images
3D T₁-w SPGR: 256 x 256 x 256 (0.78 x 0.78 x 0.78 mm³)
3D T₁-w SPGR: 256 x 256 x 256 (0.78 x 0.78 x 0.78 mm³)
Dynamic Contrast Enhanced MRI

Retrospective undersampling
3D $T_1$-w SPGR: $256 \times 186 \times 10$ ($0.93 \times 0.93 \times 3.0 \text{ mm}^3$)
35 time frames, 10 s temporal resolution
$l_1$-SPIRiT with 4D wavelet, 3D finite differences, dynamic constraints
Dynamic Contrast Enhanced MRI

Retrospective undersampling
3D T₁-w SPGR: 256 x 186 x 10 (0.93 x 0.93 x 3.0 mm³)
35 time frames, 10 s temporal resolution
l₁-SPIRiT with 4D wavelet, 3D finite differences, dynamic constraints
General Thoughts

• We’re going to see a lot more compressed sensing/constrained reconstruction in the future

• Vendors are watching the field very carefully

• Currently challenging to implement in a robust and reliable manner

• Collaborations are needed
General Thoughts (fMRI)

Currently:
- Acquire data
- Spatial and temporal smoothing
- Register
- Impose hemodynamic model
- Test for statistical significance

Potentially:
- Acquire data (appropriate sampling pattern)
  - Regularize for: spatial smoothness, temporal drift
  - Impose hemodynamic response
- Reconstruct parametric maps
Questions?